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### **Problem Solving: How do In-Service Secondary School Teachers of Mathematics Make Sense of a Non-Routine Problem Context?**

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# Problem Solving: How do In-Service Secondary School Teachers of Mathematics Make Sense of a Non-Routine Problem Context?

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## Abstract

The concept of mathematical problem solving is an important mathematical process in mathematics curricula of education systems worldwide. These math curricula demand that learners are exposed to authentic problems that foster successful problem solving. To attain this very important goal, there must be mathematics teachers well versed in content and the pedagogy of problem solving. This study investigated problem solving process of in-service secondary school teachers in a non-routine problem context. Teachers' written responses were examined based on Polya's problem solving theory to elucidate their disposition in relation to the problem context. Findings suggest that the in-service teachers exhibit (1) greater lack of understanding of the non-routine problem, (2) insufficient capacity to select appropriate heuristic strategies, and (3) total failure to reach the final "look back" stage. This study recommends in-depth examination of the role of keywords, prior knowledge and experience in mathematical problems, and the importance of written testimonies in metacognition.

## Introduction

The world over, Mathematics education curricula have recognized that "problem solving is a basic skill needed by today's learners" (Foshay & Kirkley, 2003, p.1). Jonassen (2010) considers problem solving as the "most important cognitive goal of education (formal and informal) in every educational context" (p. 2). Consequently, one of the objectives of secondary mathematics education in Kenya is for a learner to "apply mathematical knowledge and skills to familiar and unfamiliar situations." To attain this objective, Mathematics curriculum is tasked to provide learners with contexts that instill problem solving skills. Mathematics is offered as compulsory subject to learners in primary and secondary schools in Kenya. The main enabler of mathematical learning is the mathematics teacher who is well educated in mathematics content and pedagogical knowledge. Teachers of mathematics must be competent in the knowledge of mathematics subject matter and the methodology for teaching it.

Problem solving is one of the essential mathematical processes. Learners should be exposed to the skills of solving mathematical problems in familiar and unfamiliar situations. Aydogdu and Kesan (2014) observe that problem solving is the work of establishing a correlation between the things given and those requested. According to Mousoulides, Sriraman and Christou (2007), distinguishing between problem solving activity and solving traditional word problems is inevitable. The traditional word problems are equated to familiar situations while the non-routine problems are equated to unfamiliar ones. Pehkonen, Naveri and Laine (2013) provide a detailed distinction between the two by referring to standard and non-standard tasks:

If the individual can immediately recognize the procedures needed, the situation is a *standard task* (or a routine task or exercise). The term *non-standard task* is often used in reference to a task that one cannot usually find in mathematics books (p. 10). Furthermore, traditional word problems (or routine problems) are considered a representation of "simplified forms of decontextualized world based situations" (Mousoulides, Sriraman & Christou, 2007, p.24). Therefore, mathematical problem solving can be defined as a process or activity seeking to provide solutions to non-routine or non-standard tasks. According to Schoenfeld (1992), problems as routine exercises are organized to provide practice on a particular mathematical technique that, typically, has just been demonstrated to the student in class. Routine exercises follow roughly the following structure: (a) A task is used to introduce a technique; (b) The technique is illustrated; and (c) More tasks are provided so that the student may practice the illustrated skills.

Mathematical problem solving can also be considered either as a *means* or an *end* (Lester, 2013). Problem solving has been viewed as a means through which mathematics concepts, processes, and procedures are learned, that is, teaching *via* problem solving. Problem solving has also been seen as an end result of a mathematical instruction, implying teaching for problem solving, a utilitarian view of mathematical problem solving.

However, most problem solving research in mathematics education has focused primarily on *word problems* of the type emphasized in school textbooks or tests. This is where “problems” are characterized as activities that involve *getting from givens to goals when the path is not obvious* (English, Lesh, & Fennewald, 2008). Lester (2013) points out that mathematical problem solving has often been subjected to simplistic conceptualizations. Most problems as encountered in formal educational contexts are well-structured, presenting all elements of the problem and engaging limited number of rules and principles (Jonassen, 2010). There is urgent need to move away from the belief that mathematical problems are merely “another set of word problems or questions from the mathematics text book.”

### How is Problem Solving Conceptualized?

In this article, mathematical problem solving is conceptualized to comprise two levels or “worlds”: the everyday world of problems and the abstract world of mathematical concepts, symbols and operations (Lester, 2013). This view merged with Bruner’s modes of knowledge representation, can give a clear meaning, representation and process of mathematical problem solving (Figure 1).

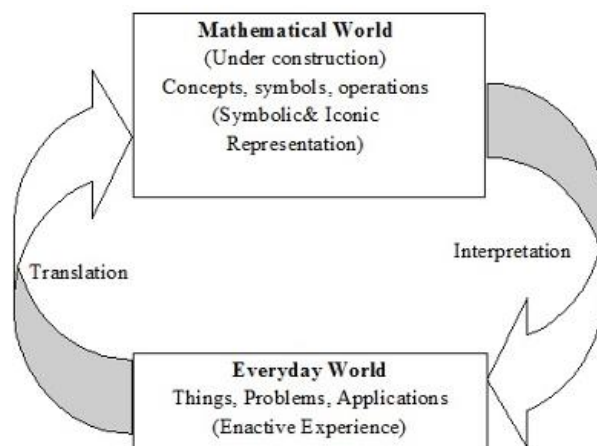


Figure 1. Problem solving process

In the mathematical world, the problem solver translates the everyday/real world problem into “easily understandable form” (Mataka, Cober, Grunert, Mutambuki & Akom, 2014, p. 165) comprising of iconic and/or symbolic representations. These modes of representations afford the solver strategies to find a mathematical solution to the problem. To find a mathematical solution, several stages are undertaken. The pioneering work of Polya gives four main (and general) stages in problem solving process. According to Polya, solving a problem involves:

1. *Understanding the problem*: In this stage, problem solver is expected to encounter the problem in a general form. The solver reads the problem thoroughly and ensures that it actually makes sense to him.
2. *Devising a plan*: At this stage, the solver thinks of and formulates different strategies for solving the problem. Some strategies (or a combination of them) will be more suitable and efficient than others. A strategy often depends on how the solver understood and interpreted the problem.
3. *Carrying out the plan*: The solver is expected to look at the listed strategies during the “devise a plan” stage. The solver has the privilege of a number of strategies to select from. This selection is an individual judgment based on the solver’s own disposition to handle a strategy with ease, effectiveness and utmost efficiency.
4. *Looking back*: This step demands that the accuracy and efficiency of the tentative solution is verified and tested. Weaknesses and strengths are identified in order to improve the strategy and to be used in other familiar or unfamiliar problem situations.

Any problem solving activity is amenable to several strategies (Aydogdu & Kesan, 2014; Lester, 2013). Upon successfully finding a mathematical solution, the solver then interprets this solution with respect to the original real world problem. Interpretation is the only way to judge the utility of the problem solving process: is the solution reasonable and useful?

Problem solving is governed by cognitive, metacognitive and affective processes (Doyle, 2005; Lester, 2013; Mayer, 2004; Mataka et al., 2014). In attempting to solve problems, problem solvers engage in a number of cognitive processes such as:

1. The need to translate each sentence into a mental reorientation,
2. The need to integrate the information to form a mental representation of the whole problem not just parts of it,
3. Planning a solution and monitoring or tracking its progress during the problem solving process, and
4. Carrying out the solution procedure (Mayer, 2004).

The Mayer (2004) cognitive processes demonstrate a problem solving paradigm that operate within the three modes of representations (between the two worlds: real and mathematical) with dynamic interaction among these representations. This dynamic interaction portends moving to and fro the activities in each stage when a solver is solving a problem. Notice that elements in the mathematical world can be manipulated as opposed to those objects (or enactive experiences) in the everyday world.

For successful mathematical problem solving in schools, teachers must be adept in their pedagogical and content knowledge, under the following:

- Designing and selecting tasks and activities;
- Listening to and observing students as they engage with problem solving activities;
- Making sure that instructional activities remain problematic for students;
- Focusing on methods students use to solve problems and being familiar with problem solving methods (e.g., heuristic strategies) that are accessible to students, and
- Being able to tell the right thing at the right time (Lester, 2013, p. 262).

In mathematical problem solving, the need for pedagogical and content knowledge are necessary (Kramarski, 2009; Lester & Kohle, 2003; Schoenfeld, 2013). Teachers of mathematics have the responsibility to provide learners with authentic and challenging problems, allowing learners through solving these problems moments to experience success. Learners essentially become astute in solving problems if they were exposed to authentic and challenging/complex problems while at the same time being able to solve some of these problems successfully. Bicer, Capraro and Capraro (2013) posit that quality of mathematical tasks or problems is determined by its “cognitive complexity as opposed to its difficulty” (p. 362). To understand the differences between cognitive complexity and difficulty, the following synopsis is quoted:

If a problem is difficult, the solution requires much effort. The extended duration of a solution (e.g., moving 500 boxes from one room to another) makes a task difficult not complex. There exists no question about how to solve a difficult task. If a problem is complex, then the solution is complicated in structure. Alternatively, from the previous example, there is no immediate procedure available to solve a complex task (e.g., arranging 500 boxes to fit into a limited space) (Bicer et al., 2013, p.362).

## Method

### Research Questions

The study set to answer the following questions:

1. Can the steps to problem solving as presented by Polya be observed in a non-routine problem context?
2. Can written explanations by solvers during problem solving provide insight into their thinking process and solution strategies?
3. What factors determine the success in problem solving?

### Sample and Sampling Techniques

The sample of respondents was obtained from in-service teachers of Mathematics who were pursuing a degree in education with mathematics as one of the teaching subjects. For these teachers, it was a requirement that they

study and pass a compulsory mathematics methods course where one of the major components is “instructional models” which includes mathematical problem solving. These in-service teachers had diplomas in education and were teaching in secondary schools. The total number of the in-service teachers was small (17), therefore, the researcher made use of all of them. The problem presented and solved was not used as part of awarding credit to the respondents.

### Instrumentation

One problem was given to the respondents to solve. This problem has been presented by Lesh and Zawojewski (1992) in their chapter on problem solving (adapted from the 1976 work of Bell, Fuson, and Lesh). The respondents were given the following instructions: *Answer the following question clearly stating the steps or procedure followed and any assumptions made.*

Two glass jars were sitting on a table. One contained 1 000 blue beads and the other contained 500 yellow beads. A teacher took 20 beads out of the blue bead jar and put them into the yellow bead jar. Then she shook the yellow bead jar until the yellow and blue beads were thoroughly mixed. Next, she randomly selected 20 beads from the yellow bead jar and put them into the blue bead jar. Are there more blue beads in the yellow bead jar than there are yellow beads in the blue bead jar?

### Results

No attempt is made in this section to provide an exhaustive presentation of all solutions. Five sample solutions which capture the essence of the problem solving process are presented. Majority of the solutions were more or less similar to these typical sample solutions. Of worthwhile concern is that only one respondent (sample solution 1) made the correct conclusion about the problem situation despite what it occurred initially as “having started in the wrong direction.”

### Sample Solutions

In the sample solutions presented:

1. Attempts are made to itemize the steps of the solutions presented by the participants. This is because some of the participants did not itemize or number their steps. For those who numbered their steps, it was realized that some of these steps were “large” and can be broken down further to assist in the interpretation of the solution. However, no changes were made on the flow or the wordings and arguments of the solver.
2. In transcribing the handwritten work of the participants, some phrases, statements or words are italicized to indicate emphasis as per author judgment. These emphases point out where the solver has strong point that was either not pursued further or having misinterpreted the requirements or their own works.
3. Attempts are made to draw diagrams the solvers presented basically to capture the main elements that are of interest in this article.

#### Sample Solution 1

1. Blue jar now contained 980 blue beads. Yellow contained 520 beads.
2. 520 beads in the yellow jar consisted of mixture of 20 blue beads and 500 yellow beads. On random picking of 20 beads from the yellow jar and putting them to the blue jar, then the blue jar had 1000 beads while the yellow jar now contained 500 beads.
3. Whether there are *more blue beads in the yellow bead jar* than there are *yellow beads in the blue bead jar depends on which beads were picked during the random picking from the yellow jar*. This is due to the fact that there are *various probabilities in which the beads could have been picked*.
4. Assumptions
  - (a) All the beads were identical only colour was different so that the teacher could not sense which bead she picked
  - (b) The 20 beads picked from the yellow bead jar are picked once randomly
5. Calculation
  - (a) There is probability that out of the 20 beads, *all the 20 beads were blue*
    - (i)  $P(20 \text{ blue beads}) = 20/520 = 0.0385$

- (ii) If this happened, automatically there will be neither blue bead in the yellow jar nor yellow in the blue jar hence there will not be more blue beads in the yellow bead jar than there are yellow beads in the blue bead jar
- (iii) *However, this probability is very low hence it is unlikely to arise.*
- (b) There is probability that out of the 20 beads picked, *all 20 beads were yellow.*
  - (i)  $P(20 \text{ yellow beads}) = 500/520 = 0.962$
  - (ii) 480 yellow beads and 20 blue beads in the yellow bead jar and 980 blue beads and 20 yellow beads in the blue bead jar.
  - (iii) Number of blue beads in the yellow bead jar and the yellow beads in the blue bead jar will be the same.
  - (iv) There will not be more blue beads in the yellow bead jar than there are yellow beads in the blue jar as there will be equilibrium. *This is possible probability since  $500/520 = 0.962 \approx 1$*
- (c) The other probability is where there will be a mixture of the 20 beads picked from the yellow bead jar; however, there are various outcomes which will arise in this mixed picking, as in table below (Table 1).

Table 1. Results when the 20 beads are randomly picked from the yellow bead jar

No. of beads picked from the yellow jar		No. of beads in the jars after picking in the jars				Are there more blue beads in the yellow beads jar than there are yellow beads in the blue bead jar
Yellow Beads	Blue Beads	Blue Jar		Yellow Jar		
		Yellow Beads	Blue Beads	Yellow Beads	Blue Beads	
20	0	20	980	480	20	No, they are equal
19	1	19	981	481	19	No, they are equal
18	2	18	982	482	18	No, they are equal
17	3	17	983	483	17	No, they are equal
...	...	...	...	...	...	...
2	18	2	998	498	2	No, they are equal
1	19	1	999	499	1	No, they are equal
0	20	0	1000	500	0	No, they are equal

#### Researcher's notes on sample solution 1

*This respondent did not draw any diagrams.* The respondent employed two strategies: probability model and drawing a table. The probability model is first referred to in step 3, that the number of blue beads picked is attributed to "various probabilities". However, to understand this strategy as used we go to step 5. Both 5(a) (i) and 5(b) (i) show probabilities calculated. Steps 5(a) (ii) and 5(b) (ii) do *not follow from* 5(a) (i) and 5(b) (i) respectively, although the participant made correct conclusions. One is therefore left to wonder "with 5(a) (ii) and 5(b) (ii) was there need then for 5(a) (i) and 5(b) (i)?" Steps 5(a) (iii) and 5(b) (iv) relates directly to 5(a) (i) and 5(b) (i) but *instantiating* these probabilities and rounding off spoils for the actual probability values obtained and the correct conclusion that had been made for the two instances. For the probability strategy, the participant made inconsistent conclusions. In the second strategy (used sequentially with the probability strategy, the participant drew a table in step 5(c) (Table 1). Drawing a table was the most successful strategy for the participant who was able to make correct conclusions.

#### Sample Solution 2

1. Draw two jars blue and yellow.



Figure 2. Two bead jars on the table

2. When 20 blue beads from the blue beads jar are taken and put into yellow; two things happened:

- (a) (i) An increase of 20 beads, blue in colour in the yellow bead jar and (ii) A decrease by 20 beads in the blue bead jar  
 (b) The diagram below (Figure 3) shows what happens when yellow bead jar is now shaken, until the beads mix thoroughly.

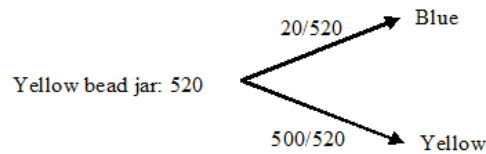


Figure 3. Probability tree diagram

- (c) By further simplification we get:  $P(B) = 20/520 = 1/26$  and  $P(Y) = 500/520 = 25/26$   
 (d) This shows that in every 26 beads picked from the yellow bead jar only one is blue and 25 are yellow. *Also it cannot be guaranteed that in every selection it will be one blue and 25 yellow but one can pick or select two yellow or even none.*  
 (e) The ratio of blue beads to yellow beads is 1:25 which is *dominated by the yellow beads*.  
 3. Now if the teacher selects 20 beads randomly from the yellow bead jar but from the above diagram (Figure 2) it shows that there are 980 blue beads and the probability of picking blue bead and yellow bead is given by  $p(B) = 1/26$  and  $p(Y) = 25/26$ .  
 4. This shows that yellow more beads were picked than the blue and taken to the blue bead jar. When the 20 beads are again put into the blue jar the total number of beads will be 1000. *Since there were 980 blue beads, this shows that there are more blue beads in the blue bead jar.*  
 5. *To conclude even if the total number of beads selected is yellow they will be only 20 out of 1000 which is still very few compared to the blue beads.*

#### Researcher's notes on sample solution 2

In step 1, this participant used “draw a diagram” strategy. This strategy however had insufficient information that would lead to any meaningful progress toward solving the problem. The strategy *just represented* the enactive experience (the real world problem) into iconic/pictorial representation without any indication of the processes of picking beads from one jar to the other and vice versa. In step 2(b) and (c), the participant tried a “probability modeling” strategy. This strategy led to misinterpretation of probabilities (25/26 and 1/26) in step 4 to mean the actual numbers of beads. Step 2(d) and (e) depicts what can be termed as “using simpler problem” strategy a corollary of the probability modeling strategy: that out of 26 beads, one would be blue and 25 would be yellow and representing a ratio 1:25. Steps 4 and 5 indicate misinterpreted requirements that led to incorrect conclusion.

#### Sample Solution 3

- Put and identify the two glass jars on the table (Figure 4)

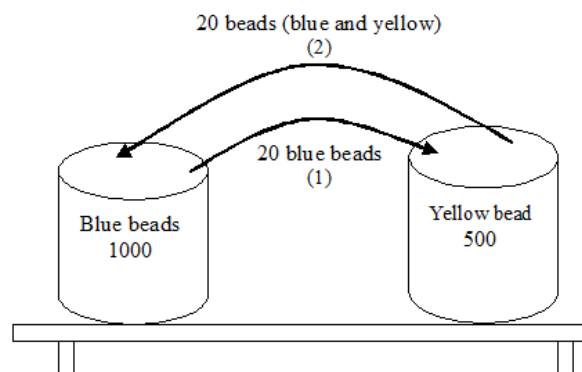


Figure 4. Jars on table showing the action of “picking”

- Take 20 blue beads from blue bead jar and put into yellow jar. Take blue beads jar now remain with 980 blue beads. The yellow beads jar now has 520 beads, i.e., 500 yellow beads and 20 blue beads.



3. Randomly select 20 beads i.e., *a mixture of blue and yellow beads* from the yellow beads jar and put into the blue beads jar. The yellow beads jar now remains with 500 beads, i.e., *a mixture of blue and yellow beads*. The blue beads jar has 1000 beads, i.e., *a mixture of blue and yellow beads*.
  4. Using probability concept:
    - (a) The probability of selecting yellow beads from the yellow bead jar is  $500/520 = 25/26$
    - $\therefore$  Out of 20 beads selected randomly, the yellow beads are  $25/26 \times 100 \approx 19.22 \approx 19$  yellow beads.
    - (b) The probability of selecting blue beads from the yellow beads jar is  $20/520 = 1/26$
    - $\therefore$  Out of the 20 beads selected randomly, the blue beads are  $= 1/26 \times 20 = 0.769 \approx 1$  blue bead.
  5. For every 20 beads randomly selected from yellow beads jar; and taken to blue beads jar, 19 beads are yellow and 1 is blue.
    - $\therefore$  The blue beads jar now has 980 blue bead + 1 blue bead returned = a total of 980 blue beads *and 19 yellow beads*
- And the yellow beads jar now remains with 500 beads a mixture of 19 blue beads that remain and 481 yellow beads that remained.
6. Conclusion. There are 19 yellow beads in the blue beads jar and there are 19 blue beads in the yellow beads jar.
- From mathematical probability, the same number of yellow beads is in the blue beads jar like the blue beads in the yellow beads jar, i.e., 19 beads each.

#### Researcher's notes on sample solution 3

The participant started with "draw a diagram" strategy (step 1). This diagram went a step further than one in Sample solution 1 by illustrating the "processes of picking" the beads from either jar. In step 3, the participant made an *erroneous assumption* (italicized) by asserting "a mixture of blue and yellow beads" is picked. With this erroneous assumption, the conclusion deduced is consequently in error. "Probability modeling" strategy was introduced in step 4 which was carried through step 5 and 6. The participant gave a probability instance and obtained approximate values of beads as 19 yellow and 1 blue. With this single instance, the participant made a correct conclusion but prematurely generalized this single case. There are no other possible cases in this strategy that could lead to sound generalization of this conclusion.

#### Sample Solution 4

1. [A diagram drawn as in Figure 2]. When 20 blue beads from the blue bead jar are put into yellow jar: 20 beads (blue) in yellow (increase) and 20 beads (blue) less in blue jar.
2. However, when the yellow bead jar is now shaken until the beads thoroughly mixed, a tree diagram will help us identify the ratios of their mixtures [A probability tree diagram as in Figure 3].
3. Simplifying:  $p(B) = 20/520 = 1/26$  and  $p(Y) = 500/520 = 25/26$ . That is, the probability of a bead being blue or yellow.
4. This implies that there are twenty five yellow beads in every twenty six beads selected and only one is blue *the same number of selection*. However, it should not be [taken] for granted that in every selection of twenty six beads, there is only one blue and twenty five other yellow but one should know that even if there are fewer blue beads than are yellow in the jar more than one may be selected in every twenty six selections, or even none will be picked in the same way. *But generally the ratio of blue beads to yellow beads in the yellow jar is 1:25*, thus the jar [yellow] is dominated by the yellow beads.
5. Note 980 blue in the blue bead jar. From 4 above, thus, [more] yellow beads were picked to the blue bead jar than the blue beads. However, in the blue bead jar, already there are 980 blues and it implies that there are going to be more blues than yellow. When 20 beads are added, the total number goes back to 1000 but majority being blue.

#### Probability of the majority and minority

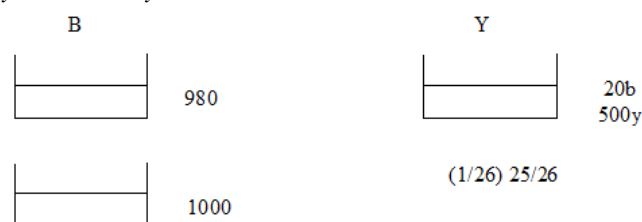


Figure 5. Jars showing beads and some probability values



6. When the beads were thoroughly mixed and then 20 beads were taken randomly out of the yellow bead jar to blue bead jar, the probability of the bead being blue is  $20/520$ , i.e.,  $1/26$ , thus very few beads were returned to the original container, the blue bead jar. The yellow beads taken to the blue bead jar have probability of  $500/520$ , i.e.,  $25/26$ , thus more yellow beads were taken to the blue bead jar container. On the other hand, the probability of a bead being yellow in the blue bead jar is  $20/1000$  thus  $1/50$ .
7. In conclusion it is seen that there are more blue beads in the yellow jar than there are yellow beads in the blue beads jar. *Compare  $1/26$  and  $1/50$ .*

#### *Researcher's notes on sample solution 4*

The participant employed the “draw a diagram” strategy. In this diagram, the processes of picking beads from either of the jars were not shown. In step 3, the participant opted for a probability modeling strategy using a tree diagram. Interpreting the probability in step 3 led to simpler problem strategy (i.e., picking 26 beads results to 25 yellow and one blue). This is followed by an erroneous assumption (italicized) “the same number of selection”, which in itself is not clear what it meant. The second argument is correct acknowledging the possibility of a mixture of blue and yellow beads in any selection. Another erroneous assumption (italicized) was made thus, “but generally the ratio of blue beads to yellow beads in the yellow jar is 1:25.” In step 6, the participant compared the probability of blue beads in the yellow jar ( $1/26$ ) with an *erroneous probability* of yellow beads in the blue bead jar ( $1/50$ ) [assuming that all the beads taken to blue bead jar are yellow!] leading to incorrect conclusion in step 7.

#### *Sample Solution 5*

1. [A diagram drawn as in Figure 4].
2. If the 1000 blue were thoroughly mixed with the 500 yellow beads, the probability of picking a blue bead would be  $1000/1500 = 2/3$ . Yellow bead =  $500/1500 = 1/3$ .
3. By picking 20 blue beads and mixing with the 500 yellow beads, the probability of picking a blue bead is *less than 2*.
4. After mixing the 500 yellow beads with the 20 blue beads, the probability of picking at random a yellow bead is:  $500/520 = 0.961538461$  while that of picking a blue bead is  $20/520 = 0.038461538$ .
5. Therefore, there are more yellow beads in the blue beads jar than there are blue beads in the yellow beads jar.

#### *Researcher's notes on sample solution 5*

This participant chose “drawing a diagram” and “probability modeling” strategies. None of the two strategies bore fruits – they were unsuccessful. Using a total of 1500 beads to get probabilities ( $1/3$  and  $2/3$ ) were erroneous and indicated lack of “understanding the problem.” Step 3 gives an interesting observation attributing probabilities to be greater than value 1. Step 4 brings another set of probabilities that are in complete disagreement with those in step 2. With all these incoherent strategies, the participant stood no chance of success.

## **Discussion**

In this section, attempts are made to answer the research questions as earlier posed. Each question is discussed below.

### *Research question 1*

The first question is “can the steps to problem solving as presented by Polya be observed in a non-routine problem context?” Based on the Polya problem solving model, a discussion of participants’ responses is presented.

Stage one, “understanding the problem”. Generally, very few participants seemed to have understood the problem well enough to offer a satisfactory translation (transforming an everyday world problem into a mathematical problem). From the sample solutions, attempts to understanding the problem involved listing the givens: variables and objects/operations such as, two jars, blue and yellow beads and stating some assumptions (e.g., in sample solution 1). For those participants whose solving processes were incorrect, their unsuccessful efforts is mainly attributed to (a) the negative influence of prior knowledge and experience and (b) the

misapplication of keywords. Probability concept (as prior experience) is taught in secondary mathematics education and post-secondary institutions to students of mathematics. This prior knowledge on probability appears in *all* sample solutions and did not enable solvers to attain victory in solving the problem. The problem under study was essentially *not a probability one* and thus the “noise” from prior knowledge and experience led to false signals toward probability modeling strategies. Prior knowledge and experience should have provided strong grounding to the solvers (Lester, 2013). Clearly, this situation suggests that problem solving has not gained attention it deserves in all the educational cycles as a mathematical process. Stage two, “devising a plan.” The quality of the plans that arose in most solutions directly emanated from stage one. The participants “understood” the problem and devised a probability, non-probability or a mixture (probability and non-probability: concurrent/parallel or sequential) plans. For example the entire sample solutions expose this assertion. Participants who thought of *keywords* as indicating a probability context devised a probability plan to attack the problem. Nonetheless; failure to successfully solve the problem is mainly attributed to the wrong choice of strategy. Aydogdu and Kesan (2014) argued that the most important factor influencing success of problem solving is the choice and use of appropriate strategy, “the appropriate strategy makes the problem solver think about the meaning of both problem sentence and the mathematical equation” (p.54).

Stage three is “carrying out the plan.” Obviously and oblivious of the interpretation of probability, most participants *misapplied* the concept of probability in the sense of the word. There was wide-ranging lack of distinction (or participants did not know the difference) between *probabilities* and the *events* that led to these probabilities. For instance when comparing the probabilities  $1/26$  and  $25/26$ . Only one participant (sample solution 1) used “drawing a table” which finally led to successfully solving the problem. The success of this stage is determined by the correct choice and the solver being keen in the manipulation of the strategy – not to incur any errors. Evidently, all the “drawing a diagram” strategy did not result to any worthwhile solution. Finally, in stage four “looking back.” It is evident that *no* participant tried to “make sense” of their answer in relation to the problem given. Perhaps, the difficulty in the interpretation of the answer in light of the original problem contributed immensely to lack of a demonstration of this stage. If this stage is adhered to by a problem solver, it can inform them of any incorrect steps that could lead to failure.

### Research Question 2

The second question is “can written explanations by solvers during problem solving provide insight into their thinking process and solution strategies?” The detailed description of steps or actions taken to solve the problem as given by the respondents portrayed their thinking processes and the choices made in solution strategies. Mataka et al. (2014) assertion that “cognitive psychologists have been interested in investigating the mental processes involved when individuals learn and solve problems” (p. 165) confirms the need to get a glimpse of what goes on in the mind of solvers during problem solving. This is often done by verbal or written testimonies. Schoenfeld (2013) considers this feature as “individual’s monitoring and self-regulation (an aspect of metacognition)” (p. 11) as one of the necessary and sufficient problem solving activity. For instance, the probability modeling strategy as evident in all the sample solutions, shed light into the misconceptions and “gaps” that existed in the thinking processes of the respondents – there is a strong disconnect between the present problem and solvers’ past experiences. Respondents actions are seen through statements made and mathematical workings displayed throughout the entire sample solutions.

### Research Question 3

The third and last question is “what factors determine the success in problem solving?” Although success or failure in mathematics problem solving is determined by a number of factors (intrinsic and extrinsic) directly or indirectly, choices made in respect to any stage (and particularly choice of strategies, also known as heuristic strategies) in solving the problem are paramount (Lester, 2013; Schoenfeld, 2013). Majority of the respondents did not “understand” the problem. This is demonstrated by the fact that all participants devised a ‘probability model’; they understood key words/phrases to point to a probability problem – this can be attributed to inappropriate use of past/prior experiences. The choice of strategies was also inappropriate. Drawing basic diagrams to represent the ‘knowns’ in the problem without actionable operations does not lead to success of any kind. Overall, majority of the participants used “drawing a picture or a diagram” strategy. For instance, the diagrams in the sample solutions (2, 3, 4 and 5) were just used to provide visual representation of how the solver mentally perceived the givens of the problem. A number of participants unsuccessfully tried to accompany these drawings/pictures by an interpretation of the requirements (the unknowns and the procedure). This observation supports Nunokawa’s (2006) conclusion that lack of thoughtful choices of a recipe of strategies such as drawings – taught or otherwise – to any kind of problem “cannot be automatically helpful” (p.52).

## Conclusion

All the participants were able to articulate their “thinking” by writing their thoughts in the process of solving the problem. This implies that writing the reasoning behind any actions in a problem solving process is useful in the understanding of what goes on in the mind of a problem solver that would otherwise escape verbal articulation (Bicer et al., 2013). It is also important as a tool for metacognition, the process of understanding ones learning. In this case, the problem solver will benefit by critically monitoring her mental processes and skills in solving mathematical problems. Success or failure in problem solving is inherent in the choices the solver makes in each stage of solving a problem. These choices determine success or failure *going forward* towards a solution. Thinking through choices that have been made can inform the solver of imminent weaknesses in all the subsequent stages of solving the problem.

None of the respondents seemed to have reached the fourth stage (looking back). In fact, all of them considered “finding an answer” as the ultimate goal. Therefore, the quality of these answers were never tested or questioned by being interpreted with respect to the original problem. Lack of success in reaching this final stage portends *the wide spread* failure to arrive at the correct answer. By looking back, the problem solver gets a “second chance or thought” to verify whether the *whole* process of solving the problem was correct and appropriate.

Majority of respondents *erroneously* interpreted a single instance of a probability value (e.g.,  $1/26$  or  $25/26$ ) and by obtaining (approximate) number of beads (1 and 19) to mean a generalized conclusion. The given mathematical problem presupposed to solvers (as in Sample solution 1) to try a number of scenarios to arrive at a plausible generalization. Because *random selection* from a set of items or objects (e.g., beads) does not guarantee certainty, this problem required solvers to *generate all possibilities* in case of the selection of 20 beads from the yellow bead jar! The Sample solution 3, presents a conclusion (generalization) in step 6 which is *correct but coming from a faulty or the misapplication of probability concept in steps 4 and 5*. For how this faulty “reasoning” led to a correct single instance is a matter of coincidence. This is because other than having “19 yellow beads in the blue bead jar and 19 blue beads in the yellow jar”, this approach *cannot yield any further possibilities*.

Although the “mixing of strategies” (e.g., probability and non-probability) did not seem to have yielded positive results, it is important to appreciate situations where solvers tend to mix strategies. The mixing of strategies could either be concurrent or sequential. One major observation is that solvers *did not* recognize the *boundary* between the two strategies. While in this article, this mixing has been referred to as either concurrent or sequential, they often overlapped. Sometimes one strategy will be left *silent without further reference* and the next strategy starts without signaling whether it is a continuation or not (refer to solution sample 1, for instance).

Using *keywords or phrases* to interpret and translate a problem does not always assure success. For routine problems, keywords play a major role in their use in the “traditional way” and thus have meanings as were taught in class. In the problem under investigation, two phrases may have suggested to the solvers to plan a probability strategy. These phrases were “beads in jars” and “random selection”. The phrase “beads in jars” should not have brought any problem but its usage in teaching probability in classrooms (other than use of urns/baskets and dice) indicate lack of creativity inherent in “textbook teaching” by teachers to use other objects to teach these concepts is lamented. Random selection is used in probability just like in this case to indicate *fairness or equal chance* in the process of selection. This would enable the solver to think of *possibilities* in drawing from a sample of items or objects.

## Recommendations

A number of recommendations to accompany these findings are presented below:

1. A careful examination of keywords in mathematical problems is necessary in order to identify inherent weaknesses in employing this strategy in solving problems. Keywords may be a necessary (and sufficient) arsenal in the toolkit of a “routine problem solver.”
2. The need for appropriate prior knowledge and experience. Prior knowledge and experience in handling routine problems or exercises (as presented in the textbooks) instill in the to-be problem solvers limited and non-transferable skills to tackle non-routine problems. Problem solving assumes that for success, prior knowledge and experience predisposes solvers to be “creative and innovative” in handling emerging problems. Therefore, pre-service and in-service education and training of teachers in mathematical problem solving should be a requirement.

3. Widespread campaign to call for “textbooks to enhance problem solving” should be advocated. This is mainly because teachers become victims of textbook teaching and hence their students will not experience “authentic problem situations”.
4. Problem solvers should be encouraged to articulate their “thinking process” during problem solving through verbal or non-verbal means. This study made use of written “testimonies.”
5. There is need to expose learners to a number of strategies with a large number of non-routine and authentic problems for them to discover the effectiveness of the different strategies in order to avoid (or minimize their) misapplication to problem situations. To be a successful mathematical problem solver, solvers must possess plenty of relevant prior content knowledge, past experience and proficiency in using a variety of representations (Lester 2013).

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